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HIGHER LOCAL QUANTUM CONSERVED CURRENTS IN TWO-DIMENSIONAL SCALAR SUPERSYMMETRIC MODELS

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The class of two-dimensional scalar supersymmetric models with non-derivative self-interactions is investigated in the context of searching for higher local quantum conservation laws. Using Zimmermann's normal product algorithm in the explicitly supersymmetric formulation, it is shown (at least for weak coupling) that the supersymmetric sine–Gordon model is the only completely integrable model in the above class. The whole infinite set of higher local conserved quantum currents of the latter is constructed. The value $\beta^2 = 4\pi$ of the supersymmetric sine–Gordon coupling constant is shown to be critical in a sense analogous to the case $\beta_{SG}^2 = 8\pi$ in the usual sine–Gordon model.

1. Introduction

In the last few years two-dimensional completely integrable field-theory models came under extremely intensive investigation (for a review see, e.g., [1]). The interest in them is mainly due to their exact solvability (spectra and S-matrices) as well as to the deep analogy between some of the most interesting among them (non-linear sigma model [2], CP^N chiral field models [3]) and the realistic four-dimensional gauge theories. In order to include interactions with fermions naturally, supersymmetric generalizations of the sine–Gordon (SG) [4, 4a], non-linear sigma [4, 5] and CP^N chiral field models [6] were proposed. On the other hand the supersymmetric sine–Gordon (SSG) model turns out to be equivalent on the quantum level to the $N = 3$ Gross–Neveu model [7]. In ref. [8] the exact quantum S-matrices of the SSG (without the kinks) and $O(N)$ non-linear sigma models were found following the general methods developed in [9].

The main ingredient of the exact solvability of two-dimensional completely integrable models is the existence of infinite sets of higher local quantum conserved currents (HLQCC) which imply non-trivial restrictions on the dynamics: absence of multiparticle production and factorization [10, 11] of all scattering processes. In particular, from the first non-trivial HLQCC the so-called factorization equations for two-particle amplitudes follow which lead to the exact determination of the latter [9].

However, the local field algebra in arbitrary quantum field models (in the present context understood as the formal algebra of Zimmermann's normal products of composite operators [12]) acquires a new structure in comparison with the corresponding classical one. Namely, due to non-multiplicative renormalizations of composite local operators, quantum equations of motion (QEM) do not in general resemble the analogous classical ones (quantum "anomalies") and therefore the powerful formalism of the inverse scattering method in classical relativistic completely integrable models [13, 1, 2] cannot be applied directly on the quantum level*. Thus proper construction of HLQCCs appears as a first important step to the exact solution.

In refs. [15, 16] one possible approach for finding HLQCCs based on the BPHZ normal product formalism [12, 17] was proposed in the case of the SG and massive Thirring models (*cf.* also [18, 19]. It was further applied for explicit construction of the first two HLQCCs in the quantum O(N) non-linear sigma model [20]. This approach is essentially algebraic, relying only on proper QEM [17] and Zimmermann identities (ZIs) for normal products [12] and using two main features: the structure of the renormalization scheme for composite operators and Lorentz structure, symmetries and (canonical) dimensions of the latter.

In the present article our aim is to find HLQCCs for theories in the class of two-dimensional scalar supersymmetric models with non-derivative self-couplings. In sect. 2, making use of supersymmetric normal product formalism (cf. [21]), it is shown (at least in the weak coupling regime) that the SSG model is the only one in the class described above, having a first HLQCC (besides the usual spin-vector supercurrent). More exactly, the most general *formal* solution to this problem is the double SSG model with a special relation between the two coupling constants $\beta_1^2\beta_2^2=\text{const}>(4\pi)^2$ [see (27), (28)]. However, it turns out that in the SSG model the value $\beta^2=4\pi$ of the coupling constant is a critical one, in the sense that at that point the theory becomes formally scale invariant and the cosine-perturbation renders it strictly (not super-) renormalizable (see sect. 3). This phenomenon has its counterpart in the usual SG model for $\beta_{SG}^2=8\pi$ [22, 23] (moreover, non-leading short-distance singularities for $\beta_{SG}^2\geqslant 4\pi$ appear there [24] which invalidate conventionally renormalized perturbation theory). On the grounds of these facts, it is argued that the latter, more general, formal solution might not be correct. In sect. 4 adapting arguments of ref. [19] the whole infinite series of HLQCCs of the SSG

* Let us mention however, that recently a new, very interesting approach [14] to the exact solution of quantum completely integrable models appeared which is a non-trivial quantum analogue of the inverse scattering method.

model is constructed*. This provides complete dynamical justification of the proposed exact quantum S-matrix of the model [8]**.

2. Supersymmetric normal product formalism and HLQCCs

2.1. Preliminaries and notation

We shall use the standard notions of the superspace approach to supersymmetry [28] appropriately accomodated to the case of two-dimensional space-time. The general form of a scalar superfield reads

$$\phi(x,\theta)=\varphi(x)+i\bar{\theta}\psi(x)+\tfrac{1}{2}i\bar{\theta}\theta F(x)\,,$$
$$\bar{\theta}^{\alpha}=C^{-1\alpha\beta}\theta^{\beta},\qquad C^{-1}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\tag{1}$$

where $\varphi(x), F(x)$ are (pseudo) scalar fields (even elements of a Grassmann algebra), $\psi_{\alpha}(x)$, θ^{α} are two-component Majorana spinors (odd elements of a Grassmann algebra). We choose the following particular representation for the Dirac matrices:

$$\gamma_0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \gamma_1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};$$
$$\hat{p}\equiv p^{\mu}\gamma_{\mu}=\begin{pmatrix}0 & p_0-p_1\\ p_0+p_1 & 0\end{pmatrix}.$$

Light-cone coordinates and light-cone components of Lorentz vectors (and tensors) are defined as follows:

$$\xi=\tfrac{1}{2}(x^0+x^1),\qquad \eta=\tfrac{1}{2}(x^0-x^1)\,,$$
$$A_{\xi}=A_0+A_1,\qquad A_{\eta}=A_0-A_1\,,$$
$$A_{\mu}B^{\mu}=\tfrac{1}{2}(A_{\xi}B_{\eta}+A_{\eta}B_{\xi}),\qquad \text{etc}\,.$$

The supersymmetric covariant derivative $\mathcal{D}=\partial/\partial\bar{\theta}-i(\hat{\partial}\theta)$ has components of the form

$$\mathcal{D}_1=\partial_{\theta^2}-i\theta^2\partial_{\eta},\qquad \mathcal{D}_2=-\partial_{\theta^1}-i\theta^1\partial_{\xi}\,,$$
$$\mathcal{D}_1\mathcal{D}_1=-i\partial_{\eta},\qquad \mathcal{D}_2\mathcal{D}_2=i\partial_{\xi}\,,$$
$$\mathcal{D}_2\mathcal{D}_1=-\mathcal{D}_1\mathcal{D}_2=-\partial_{\theta^1}\partial_{\theta^2}-i\theta^2\partial_{\theta^1}\partial_{\eta}-i\theta^1\partial_{\theta^2}\partial_{\xi}+\theta^2\theta^1\partial_{\xi}\partial_{\eta}\,.\tag{2}$$

* The infinite set of classical conserved currents in the SSG model was found in [25] and subsequently the corresponding supersymmetric inverse scattering problem [26, 26a] and the associated Backlund transformation [26a] were presented.

** The first non-trivial HLQCC of the supersymmetric O(N) non-linear sigma model was recently constructed in [27] by means of a supersymmetric extension of the method of ref. [20].

Grassmann differentiation and integration are introduced according to the rules in ref. [29]:

$$\mathscr{D}_\alpha(AB) = (\mathscr{D}_\alpha A)B \pm A(\mathscr{D}_\alpha B), \tag{3a}$$

$$\int \mathrm{d}\theta^\alpha = -\partial_{\bar\theta^\alpha}, \qquad \int \mathrm{d}^2\theta \equiv \int \mathrm{d}\theta^1\,\mathrm{d}\theta^2 = \partial_{\theta^1}\partial_{\theta^2}, \qquad \int \mathrm{d}^2\theta\, f(\theta+\theta') = \int \mathrm{d}^2\theta\, f(\theta), \tag{3b}$$

where the minus sign in (3a) holds when A is an odd element. The Grassmann delta-function has the properties [30]

$$\delta(\theta-\theta') = (\theta^2-\theta'^2)(\theta^1-\theta'^1), \qquad \int \mathrm{d}^2\theta\,\delta(\theta-\theta') = 1,$$
$$\delta(\theta-\theta')f(\theta) = \delta(\theta-\theta')f(\theta'). \tag{4}$$

The Lorentz weights w of the various objects introduced so far are as follows (i.e., the latter transform under the proper two-dimensional Lorentz group by multiplicative factors $\exp(w\chi)$, χ-rapidity):

$$w(\partial_\eta) = 1, \qquad w(\partial_\xi) = -1, \qquad w(\mathscr{D}_1) = w(\partial_{\theta^2}) = w(\theta^1) = w(\psi_1) = \tfrac{1}{2},$$
$$w(\mathscr{D}_2) = w(\partial_{\theta^1}) = w(\theta^2) = w(\psi_2) = -\tfrac{1}{2}. \tag{5}$$

The langrangian of the SSG model and the corresponding classical equations of motion read

$$\mathscr{L}_{\mathrm{SSG}}(x,\theta) = \tfrac{1}{2}\mathscr{D}_2\phi\mathscr{D}_1\phi - (m/\beta^2)\cos\beta\phi\ ; \tag{6}$$

$$\mathscr{D}_2\mathscr{D}_1\phi = (m/\beta)\sin\beta\phi\,, \tag{7}$$

or equivalently in components (1)

$$\mathscr{L}_{\mathrm{SSG}}(x) = \tfrac{1}{2}\partial_\xi\varphi\partial_\eta\varphi + \tfrac{1}{2}i\psi_1\partial_\xi\psi_1 - \tfrac{1}{2}i\psi_2\partial_\eta\psi_2 - \tfrac{1}{2}F^2$$
$$- m\psi_1\psi_2\cos\beta\varphi + \frac{im}{\beta}F\sin\beta\varphi\ ; \tag{6'}$$

$$\partial_\xi\partial_\eta\varphi = m\beta\psi_1\psi_2\sin\beta\varphi + imF\cos\beta\varphi, \qquad F = \frac{im}{\beta}\sin\beta\varphi\,,$$
$$i\partial_\xi\psi_1 = m\psi_2\cos\beta\varphi, \qquad i\partial_\eta\psi_2 = m\psi_1\cos\beta\varphi\,. \tag{7'}$$

2.2. *Invariant perturbation theory and BPHZ renormalization*

From here to the end of this section we shall consider more general scalar supersymmetric models with non-derivative self-couplings:

$$\mathscr{L}(x,\theta) = \tfrac{1}{2}\mathscr{D}_2\phi\mathscr{D}_1\phi + \tfrac{1}{2}m\phi^2 + \mathscr{L}_{\mathrm{I}}(\phi)\,. \tag{8}$$

Graphical elements of supersymmetric perturbation theory [28, 30] are vertices determined by $\mathscr{L}_{\mathrm{I}}(\phi)$ and propagators:

$$D(p;\theta,\theta')=\mathrm{e}^{\bar{\theta}\hat{p}\theta'}(1-m\delta(\theta-\theta'))[m^2-p^2-i0]^{-1}\,. \tag{9}$$

Besides the usual integration over independent internal loop momenta one associates to each internal vertex an integration over the corresponding θ [30].

Ultraviolet (UV) power counting is an important ingredient in the BPHZ renormalization scheme. In the present supersymmetric context one assigns to the Grassmann elements the following UV dimensions:

$$\dim\theta^\alpha=-\tfrac{1}{2},\qquad \dim(\partial_{\theta^\alpha})=\dim\mathscr{D}_\alpha=\tfrac{1}{2},\qquad \alpha=1,2\,. \tag{10}$$

Thus the general formula for the canonical UV degree $\omega(\Gamma)$ of an arbitrary super-graph Γ of the theory (8) reads [$V(\Gamma)$ is the number of vertices in Γ]:

$$\omega(\Gamma)=2(1-\mathrm{V}(\Gamma)). \tag{11}$$

When one extra composite operator P vertex insertion is present (this is the only relevant case we shall need below), eq. (11) is modified to:

$$\omega(\Gamma)=\dim P-2V(\Gamma). \tag{11$'$}$$

In order to assure minimal UV subtractions we shall employ the trick of partially "soft" mass [31]. Namely, the mass m in the numerator of (9) is replaced by sm, where s $(0\leqslant s\leqslant 1)$ is an auxiliary parameter. Modified subtraction operators $\tau^\omega_{p,s}$ are defined by the properties [21]

$$\begin{aligned}&\tau^\omega_{p,s}f(p,s)=t^\omega_{p,s}f(p,s),\\&\tau^\omega_{p,s}\theta^\alpha f(p,s;\theta)=\theta^\alpha\tau^{\omega+1/2}_{p,s}f(p,s;\theta)\,,\end{aligned} \tag{12}$$

$f(p,s;\theta)$ being an arbitrary function and $t^\omega_{p,s}$ the standard Taylor subtraction operators of order ω in the variables p, s. After implementing all subtractions in the corresponding Zimmermann's "forest formula" [31]:

$$R_\Gamma((p),s,(k);(\theta),(\varphi))=\sum_{\mathrm{U}\in\mathscr{F}_\Gamma}\prod_{\gamma\in\mathrm{U}}(-\tau^{\omega(\gamma)}_{p,s})I_\Gamma((p),s,(k);(\theta),(\varphi)) \tag{13}$$

we set $s=1$. In (13) the following standard notation is used: R_Γ is the renormalized integrand; I_Γ is the unrenormalized integrand of the super-graph Γ; $\mathscr{F}_\Gamma$ is the set of all Γ-forests U; γ are subgraphs of Γ, elements of the U's; (p), (k), (θ), (φ) are sets of external and internal momenta, external and internal Grassmann factors, respectively. By definition $\tau^{\omega(\gamma)}_{p,s}\equiv 0$ when acting on one-particle reducible sub-graphs.

One can easily perform the internal Grassmann integrations by means of the formulas [*cf.* (3b), (4)]

$$\int \mathrm{d}^2\varphi\,\exp\left\{\sum_{l=1}^N\bar{\theta}_l\hat{p}_l\varphi\right\}=\sum_{l=1}^N p_l^2\,\delta(\theta_l)+\sum_{l,k=1,l\neq k}^N p_{l,\eta}p_{k,\xi}\theta_l^2\theta_k^1, \tag{14}$$

$$\int \mathrm{d}^2\varphi\varphi^1 \exp\left\{\sum_{l=1}^{N} \bar{\theta}_l\hat{p}_l\varphi\right\} = \sum_{l=1}^{N} p_{l,\eta}\theta_l^2 ,$$

$$\int \mathrm{d}^2\varphi\,\varphi^2 \exp\left\{\sum_{l=1}^{N} \bar{\theta}_l\hat{p}_l\varphi\right\} = \sum_{l=1}^{N} p_{l,\xi}\theta_l^1 , \tag{14 (cont.)}$$

thus reducing I_Γ to a sum of contributions $J_\Gamma^{(\Lambda)}$, each of them corresponding to a usual Feynman graph multiplied by external (Grassmann) factors of the form:

$$I_\Gamma^{(\Lambda)}((p), s, (k); (\theta)) = \exp\left\{\sum_{a,b} \bar{\theta}_a\hat{p}_{ab}\theta_b\right\} \prod_{a_1,b_1} [sm\,\delta(\theta_{a_1}-\theta_{b_1})]$$

$$\times \prod_c \theta_c^2 p_{c,\eta} \prod_d \theta_d^1 p_{d,\xi} J_\Gamma^{(\Lambda)}((p), s, (k)). \tag{15}$$

Here Λ is a shorthand notation for the corresponding configurations of indices $\{a, b\}$, $\{a_1, b_1\}$, $\{c\}$, $\{d\}$, labelling external Grassmann factors and linear combinations of external momenta. Combining (13), (15) and accounting for (12) one obtains

$$R_\Gamma((p), s, (k); (\theta)) = \sum_\Lambda \exp\left\{\sum_{a,b} \bar{\theta}_a\hat{p}_{ab}\theta_b\right\} \prod_{a_1,b_1} [sm\,\delta(\theta_{a_1}-\theta_{b_1})]$$

$$\times \prod_c \theta_c^2 p_{c,\eta} \prod_d \theta_d^1 p_{d,\xi} R_\Gamma^{(\Lambda)}((p), s, (k)),$$

$$R_\Gamma^{(\Lambda)}((p), s, (k)) = \sum_{\mathrm{U}\in\mathscr{F}_\Gamma} \prod_{\gamma\in\mathrm{U}} (-t_{p,s}^{\omega_\Lambda(\gamma)}) J_\Gamma^{(\Lambda)}((p), s, (k)) ; \tag{16a}$$

$$\omega_\Lambda(\gamma) = \omega(\gamma) + V^{\mathrm{int}}(\gamma) - \tfrac{1}{2}(|c| + |d|)$$
$$= \begin{cases} 2 - V(\gamma) - V^{\mathrm{ext}}(\gamma) - \tfrac{1}{2}(|c|+|d|) , \\ \dim P - V(\gamma) - V^{\mathrm{ext}}(\gamma) - \tfrac{1}{2}(|c|+|d|) , \end{cases} \tag{16b}$$

where $|c|+|d|$ is always even; $|c|$, $|d|$ denote the number of factors $\theta_c^2 p_{c,\eta}$, $\theta_d^1 p_{d,\xi}$, respectively; $V(\gamma) = V^{\mathrm{int}}(\gamma) + V^{\mathrm{ext}}(\gamma)$; $V^{\mathrm{int(ext)}}(\gamma)$ is the number of internal (external) vertices in γ. Eqs. (16) lead not only to the superrenormalizability of the models (8) (with the only UV divergence occurring in point-loops) but also determine the particular structure of the relevant ZIs in the QEM [see (24) and figs. 2 and 4 below] which is crucial for the existence of HLQCCs.

2.3. Quantum equations of motion

QEM for supersymmetric Green functions with composite operator vertex insertions can be derived by the standard BPHZ procedure [17] with the only exception that due to the partially "soft" mass renormalization (12) one should use a modified identity for the propagator (9) in momentum space ("$\sim$" means Fourier transform):

$$[\tilde{\mathscr{D}}_2\tilde{\mathscr{D}}_1 - sm]\left\{ e^{\bar{\theta}\hat{p}\theta'} \frac{1 - sm\,\delta(\theta-\theta')}{m^2 - p^2 - i0} \right\} = \delta(\theta-\theta') + m^2(s^2-1)\frac{\delta(\theta-\theta')}{m^2-p^2-i0}. \quad (17)$$

In this way we obtain ($\langle\ldots\rangle$ denote connected time-ordered Green functions; $Q(\mathscr{D})$ is an arbitrary differential monomial in $\mathscr{D}_\alpha$; P is an arbitrary composite operator):

$$\begin{aligned}
&\langle \mathrm{N}[P.Q(\mathscr{D})(\mathscr{D}_2\mathscr{D}_1\phi)](x,\theta)X_\phi\rangle = m\langle \mathrm{N}[P.Q(\mathscr{D})\phi](x,\theta)X_\phi\rangle \\
&\quad + \langle \mathrm{N}[P.Q(\mathscr{D})(\delta\mathscr{L}_{\mathrm{I}}/\delta\phi)](x,\theta)X_\phi\rangle - i\sum_{l=1}^{L_\phi}[Q(\mathscr{D})\,\delta(x-x_l)\,\delta(\theta-\theta_l)] \\
&\quad \times \langle \mathrm{N}[P](x,\theta)\hat{X}^l_\phi\rangle + m^2\langle \mathrm{N}[P.Q(\mathscr{D})\{(s^2-1)\phi\}](x,\theta)X_\phi\rangle\,; \\
&\quad X_\phi \equiv \prod_{l=1}^{L_\phi}\phi(x_l,\theta_l), \qquad \hat{X}^l_\phi \equiv \prod_{l'=1,\,l'\neq l}^{L_\phi}\phi(x_{l'},\theta_{l'})\,.
\end{aligned} \quad (18)$$

Here the symbol N denotes the canonical (minimally subtracted) normal product, except in the last "anomalous" anisotropic term on the r.h.s. of eq. (18). The meaning of the curly brackets there is twofold. Firstly, the propagator corresponding to ϕ inside the brackets is not (9) but a modified one (*cf.* (17)): $\delta(\theta-\theta')[m^2 - p^2 - i0]^{-1}$ (graphically this propagator will be represented by a marked line, see figs. 1–4). Secondly, one assigns to each subgraph $\dot{\Gamma}$ containing the modified propagator UV *over*subtraction degree:

$$\tilde{\omega}(\dot{\Gamma}) = \omega(\dot{\Gamma}) + 2 = \omega(\Gamma) + 1, \qquad \omega_\Lambda(\Gamma) = \omega_\Lambda(\dot{\Gamma})\,, \quad (19)$$

where Γ is topologically identical to $\dot{\Gamma}$ but with all propagators normal. The last two relations in (19) follow from the presence of $\delta(\theta-\theta')$ in the modified propagator and from (4), (12). The anisotropic normal products in QEM are to be expanded in terms of canonical ones by means of the ZIs (see figs. 1–4).

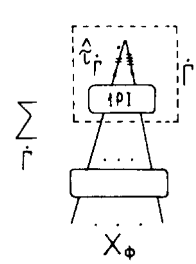

Fig. 1. Graphical structure of the ZI for $\langle \mathrm{N}[\mathscr{D}_1^4\phi\mathscr{D}_1^2\{(s^2-1)\phi\}](x,\theta)X_\phi\rangle$ (1PI ≡ one-particle irreducible).

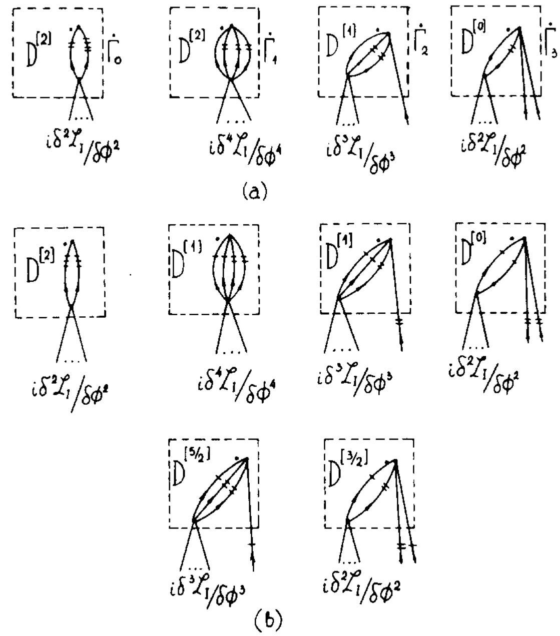

Fig. 2. Complete list of supergraphs giving rise to the ZI for the r.h.s. of eq. (23).

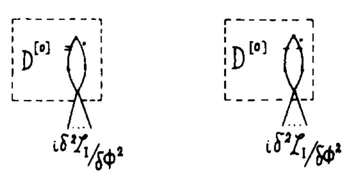

Fig. 3. Supergraphs contributing to the ZI (33).

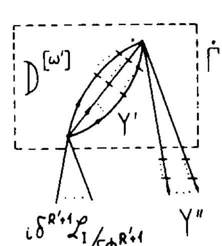

Fig. 4. General graphical structure of the ZI for $\langle \mathrm{N}[\mathscr{D}_1^{r_1}\phi \ldots \mathscr{D}_1^{r_c-1}\{(s^2-1)\phi\} \ldots \mathscr{D}_1^{r_{2R}}\phi] \times (x,\theta) X_\phi \rangle$. Here the following notations are used: $Y' = \mathscr{D}_1^{r'_1}\phi \ldots \mathscr{D}_1^{r'_{R'}}\phi$, $Y'' = \mathscr{D}_1^{r''_1}\phi \ldots \mathscr{D}_1^{r''_{R''}}\phi$, $Y'Y''\mathscr{D}_1^{r_c}\phi = Y^{(k)}_{\{(r)\}}$; $\omega' = \frac{1}{2}(r_c + 1 + \sum_{a=1}^{s'} r'_a) - 2$.

2.4. Higher local quantum conserved current

In analogy with the usual non-supersymmetric case, the first step to the establishment of complete integrability should be to look for a HLQCC which would give rise to the non-trivial conservation law

$$\sum_{\text{in}} (p_{l,\eta})^{\pm 3} = \sum_{\text{out}} (p'_{l,\eta})^{\pm 3}\,, \tag{20}$$

where the sums run over the sets of in- and out-particle momenta for an arbitrary scattering process*. Eventual fulfilment of (20) would prevent multiparticle production in two-particle collisions and would imply factorization of the three-particle S-matrix to products of two-particle ones, i.e., eq. (20) would lead to the exact determination of the two-particle amplitudes [9]. The most general admissible supersymmetric structure for such an HLQCC reads

$$\begin{aligned} J^1_{7/2}(x,\theta) &= \mathrm{N}[F(\phi)\mathscr{D}_1^3\phi\mathscr{D}_1^4\phi + G(\phi)\mathscr{D}_1\phi(\mathscr{D}_1^2\phi)^3](x,\theta) \\ J^2_{7/2}(x,\theta) &= m\mathrm{N}[A_1(\phi)\mathscr{D}_1^5\phi + A_2(\phi)\mathscr{D}_1\phi\mathscr{D}_1^4\phi + A_3(\phi)\mathscr{D}_1^2\phi\mathscr{D}_1^3\phi \\ &\quad + A_4(\phi)\mathscr{D}_1\phi(\mathscr{D}_1^2\phi)^2](x,\theta)\,. \end{aligned} \tag{21}$$

All functions in (21) (which are *a priori* arbitrary entire functions in $\phi(x,\theta)$) are to be determined by the requirement that the corresponding Ward identity (WI) for the conservation of $J^{\alpha}_{7/2}(x,\theta)$ holds:

$$\begin{aligned} &\mathscr{D}_2\langle J^1_{7/2}(x,\theta)X_\phi\rangle - \mathscr{D}_1\langle J^2_{7/2}(x,\theta)X_\phi\rangle = -i\sum_{l=1}^{L_\phi}[\mathscr{D}_1^2\,\delta(x-x_l)\,\delta(\theta-\theta_l)] \\ &\quad\times\langle \mathrm{N}[F(\phi)\mathscr{D}_1^4\phi](x,\theta)\hat{X}^l_\phi\rangle - i\sum_{l=1}^{L_\phi}\delta(x-x_l)\,\delta(\theta-\theta_l) \\ &\quad\times\langle \mathrm{N}[G(\phi)(\mathscr{D}_1^2\phi)^3](x,\theta)\hat{X}^l_\phi\rangle + i\sum_{l=1}^{L_\phi}[\mathscr{D}_1^3\,\delta(x-x_l)\,\delta(\theta-\theta_l)] \\ &\quad\times\langle \mathrm{N}[F(\phi)\mathscr{D}_1^3\phi](x,\theta)\hat{X}^l_\phi\rangle \\ &\quad+3i\sum_{l=1}^{L_\phi}[\mathscr{D}_1\,\delta(x-x_l)\,\delta(\theta-\theta_l)]\langle \mathrm{N}[G(\phi)\mathscr{D}_1\phi(\mathscr{D}_1^2\phi)^2](x,\theta)\hat{X}^l_\phi\rangle\,. \end{aligned} \tag{22}$$

One can easily convince himself that actually $F(\phi)$, $G(\phi)$ must be constants independent of $\phi(x,\theta)$. This is because terms containing $\mathscr{D}_2\phi\delta F/\delta\phi$, $\mathscr{D}_2\phi\delta G/\delta\phi$ in $\mathscr{D}_2 J^1_{7/2}$ can never be compensated by terms arising in $\mathscr{D}_1 J^2_{7/2}$. We choose $F=1$ (another choice would lead to a trivial change of normalization of the corresponding conserved charge).

* Let us recall that on-mass-shell $p_\xi = m^2/p_\eta$. Both conservation laws (20) are connected through space reflection.

Substituting (21) into (22) and using QEM (18) we obtain $[V(\phi)\equiv\frac{1}{2}m\phi^2+\mathscr{L}_{\rm I}(\phi)]$

$$\begin{aligned}&\langle \mathrm{N}[-mA_1(\phi)\mathscr{D}_1^6\phi+m(A_2(\phi)-\delta A_1/\delta\phi)\mathscr{D}_1\phi\mathscr{D}_1^5\phi-(mA_2(\phi)+mA_3(\phi)\\&\quad-\delta^2V/\delta\phi^2)\mathscr{D}_1^2\phi\mathscr{D}_1^4\phi+(2mA_4(\phi)-m\,\delta A_3/\delta\phi-\delta^3V/\delta\phi^3)\mathscr{D}_1\phi\mathscr{D}_1^2\phi\mathscr{D}_1^3\phi\\&\quad-(mA_4(\phi)-G\delta V/\delta\phi)(\mathscr{D}_1^2\phi)^3](x,\theta)X_\phi\rangle\\&\quad=-m^3\langle \mathrm{N}[\mathscr{D}_1^4\phi\mathscr{D}_1^2\{(s^2-1)\phi\}+\mathscr{D}_1^3\phi\mathscr{D}_1^3\{(s^2-1)\phi\}\\&\quad+G(\mathscr{D}_1^2\phi)^3\{(s^2-1)\phi\}+3G(\mathscr{D}_1^2\phi)^2\mathscr{D}_1\phi\mathscr{D}_1\{(s^2-1)\phi\}](x,\theta)X_\phi\rangle\,.\end{aligned}\tag{23}$$

The general graphical structure of the ZI for, e.g., the first anisotropic normal product on the r.h.s. of (23) is displayed in fig. 1. Bars on ϕ-lines denote $\mathscr{D}_1$ derivatives, the operators $\hat{\tau}_{\dot\Gamma}\equiv\tau_{p,s}^{\omega(\dot\Gamma)+2}-\tau_{p,s}^{\omega(\dot\Gamma)}$ act on the corresponding (renormalized) subgraphs $\dot\Gamma$ inside the boxes.

Now observe that $w(\dot\Gamma)=3$ for $\dot\Gamma$ in fig. 1 and that, on the other hand, from (16)

$$\begin{aligned}&\hat{\tau}_{\dot\Gamma}\int R_{\dot\Gamma}((p),s,(k);(\theta))\prod\frac{\mathrm{d}^2k}{(2\pi)^2}\\&\quad=\sum_{\Lambda}\exp\left\{\sum_{a,b}\bar\theta_a\hat{p}_{ab}\theta_b\right\}\prod_{a_1,b_1}[sm\delta(\theta_{a_1}-\theta_{b_1})]\\&\quad\times\prod_c\theta_c^2p_{c,\eta}\prod_d\theta_d^1p_{d,\xi}(t_{p,s}^{\omega_\Lambda(\dot\Gamma)+2}-t_{p,s}^{\omega_\Lambda(\dot\Gamma)})\\&\quad\times\int R_{\dot\Gamma}^{(\Lambda)}((p),s,(k))\prod\frac{\mathrm{d}^2k}{(2\pi)^2}\,,\end{aligned}$$

where the last integral has clearly Lorentz weight $w_\Lambda(\dot\Gamma)=3-\frac{1}{2}(|c|+|d|)$. Apparently the Taylor operator acting on the latter will give a non-vanishing contribution if and only if

$$\omega_\Lambda(\dot\Gamma)+2\leqslant w_\Lambda(\dot\Gamma)\qquad\text{i.e., } V(\dot\Gamma)+V^{\rm ext}(\dot\Gamma)\leqslant 2-|d|\,.\tag{24}$$

This is due to the peculiarity of two-dimensional Lorentz kinematics (the absence of invariant tensors of odd rank and the antidiagonal representation of the metric tensor $g_{\mu\nu}$ in light-cone components). From (24) we conclude that the only supergraphs $\dot\Gamma$ giving rise to the corresponding ZI are those with $V(\dot\Gamma)=V^{\rm ext}(\dot\Gamma)=1$ (of course $|d|=0$) and moreover, only the part

$$D^{[\omega_\Lambda(\dot\Gamma)+2]}\equiv t_{p,s}^{\omega_\Lambda(\dot\Gamma)+2}-t_{p,s}^{\omega_\Lambda(\dot\Gamma)+1}$$

of the operator $\hat\tau_{\dot\Gamma}$ contributes (see fig. 2). Exactly the same statement for the structure of the ZIs is valid in all other cases of interest, i.e., for the anisotropic normal products appearing in the WIs for *all* HLQCCs [see fig. 4] (*cf.* [15, 16] for the case of usual non-supersymmetric scalar models). The full list of graphs contributing to the ZI for the r.h.s. of (23) is depicted in fig. 2.

On account of the above facts eq. (23) reduces to the form

$$\begin{aligned}
&\langle \mathrm{N}[(-mA_1(\phi)+2c_0\,\delta^3V/\delta\phi^3+4Gc_1\,\delta^5V/\delta\phi^5)\mathscr{D}_1^6\phi \\
&+(-m\,\delta A_1/\delta\phi+mA_2(\phi)+3c_2G\delta^4V/\delta\phi^4)\mathscr{D}_1\phi\mathscr{D}_1^5\phi \\
&+(-mA_2-mA_3+\delta^2V/\delta\phi^2+(6c_0+9c_2G)\,\delta^4V/\delta\phi^4 \\
&+12Gc_1\delta^6V/\delta\phi^6)\mathscr{D}_1^2\phi\mathscr{D}_1^4\phi \\
&+(2mA_4(\phi)-m\,\delta A_3/\delta\phi-(1-6c_3G)\,\delta^3V/\delta\phi^3 \\
&+6c_2G\,\delta^5V/\delta\phi^5)\mathscr{D}_1\phi\mathscr{D}_1^2\phi\mathscr{D}_1^3\phi \\
&+(-mA_4(\phi)+G\delta V/\delta\phi+6c_3G\,\delta^3V/\delta\phi^3+(2c_0+9c_2G)\,\delta^5V/\delta\phi^5 \\
&+4c_1G\,\delta^7V/\delta\phi^7)(\mathscr{D}_1^2\phi)^3](x,\theta)X_\phi\rangle=0\,.
\end{aligned} \tag{25}$$

Here the constants c_k, $k=0,\ldots,3$, are the computable contributions of the box-subgraphs of fig. 2a. The contributions of the remaining box-subgraphs of fig. 2b are simply expressed in terms of c_k. Thus the WI (22) holds if and only if the coefficients in front of each independent Lorentz structure $\mathscr{D}_1^6\phi,\ldots,(\mathscr{D}_1^2\phi)^3$ on the l.h.s. of (25) vanish. This gives a set of 4 linear equations for $A_i(\phi)$, $i=1,\ldots,4$, plus the most important ordinary differential equation for $V(\phi)$:

$$6c_2G\,\delta^5V/\delta\phi^5-[1-3G(c_2+2c_3)]\,\delta^3V/\delta\phi^3+G\,\delta V/\delta\phi=0\,. \tag{26}$$

The general formal solution of (26) respecting space-reflection symmetry $\phi\to-\phi$ reads:

$$V(\phi)=-\frac{\tilde m_1}{\beta_1^2}\cos\beta_1\phi-\frac{\tilde m_2}{\beta_2^2}\cos\beta_2\phi\,, \tag{27}$$

$$\begin{aligned}
&G=-\beta_j^2[1-3\beta_j^2(c_2+2c_3)+6c_2\beta_j^4]^{-1},\qquad j=1,2\,,\\
&\beta_1^2\beta_2^2=1/6c_2>(4\pi)^2\,,
\end{aligned} \tag{28}$$

$$c_2=(32\pi^2)^{-1}\left\{\tfrac{2}{3}+\tfrac{1}{27}\pi\sqrt{3}-2\int_0^1\frac{x(1-x)\ln(1/x)\,\mathrm{d}x}{[1-x(1-x)]^3}\right\},$$

where $\tilde m_1$, $\tilde m_2$ are arbitrary parameters with dimensions of mass.

One possibility is to put $\tilde m_2=0$, $\tilde m_1\neq0$, $\beta_1\equiv\beta$ arbitrary (or *vice versa*) thus obtaining the SSG model. Clearly, proceeding exactly in the same manner on classical level we would get just the classical SSG model as the *only* possible solution because all terms containing c_k arise from purely quantum effects (i.e., they are due to the presence of "anomalous" anisotropic terms on the r.h.s. of (23)] and accordingly would be absent in the corresponding classical equations which are analogues of (25), (26).

However there are arguments which lead us to consider as unreliable the more general double SSG solution (27), (28) (i.e., $\tilde m_j\neq0$, $j=1,2$). Eq. (28) shows that both coupling constants β_1, β_2 cannot be simultaneously small and, more concretely, at least one of the latter will exceed the value 4π which according to the discussion in sect. 3 is a critical one. At that point the quantum energy-momentum tensor of the SSG model has vanishing trace (i.e., scale invariance) [see (35)] and the total anomalous UV dimension of the cosine-perturbation $\dim(\cos\beta\phi)=\beta^2/4\pi$ [see (38)] becomes equal to 1, i.e., it ceases to be super-renormalizable. Therefore the procedure of this section for deriving HLQCCs cannot be expected to work in the non-renormalizable strong coupling regime (28).

The result up to now can be summarized as follows. The SSG model with quantum effective (in the sense of Zimmermann [12]) lagrangian

$$\mathscr{L}_{\text{eff}}(x,\theta)=\tfrac{1}{2}\mathrm{N}_1[\mathscr{D}_2\phi\mathscr{D}_1\phi+m\phi^2]'(x,\theta)$$
$$-\mathrm{N}\left[\frac{\tilde{m}}{\beta^2}(\cos\beta\phi-1)+\tfrac{1}{2}m\phi^2\right](x,\theta)\,, \tag{29}$$

is the only one (at least in the weak coupling regime) in the class of scalar supersymmetric models (8) having an HLQCC. In (29) the prime on the first normal product indicates partially "soft" mass renormalization (13) and the quantity $\tilde{m}=mf(m/\mu)$ contains (finite) mass counter-term additions accounting for the freedom in subtracting the divergent point-loops, μ being an arbitrary subtraction point. The normalization of $f(m/\mu)$ is such that m is precisely the physical mass (of the fundamental particles).

To end this section let us write down the exact explicit expression for the first non-trivial HLQCC in the SSG model (*cf.* [16] for the corresponding HLQCC in the usual SG model) $[G=G(\beta^2)$ from (27)]:

$$J^1_{7/2}(x,\theta)=\mathrm{N}[\mathscr{D}_1^3\phi\mathscr{D}_1^4\phi-\beta^2(1-3\beta^2(c_2+2c_3)+6c_2\beta^4)^{-1}\mathscr{D}_1\phi(\mathscr{D}_1^2\phi)^3](x,\theta)\,;$$
$$J^2_{7/2}(x,\theta)=\mathrm{N}\Big[\frac{\tilde{m}}{\beta}\sin\beta\phi([-2c_0\beta^2+4\beta^4c_1G(\beta^2)]\mathscr{D}_1^5\phi$$
$$+[-\beta^2+2c_0\beta^4+\beta^2G(\beta^2)(3c_2+3c_2\beta^2-4c_1\beta^4)]\mathscr{D}_1\phi(\mathscr{D}_1^2\phi)^2)\Big](x,\theta)$$
$$+\mathrm{N}[\tilde{m}\cos\beta\phi([-2c_0\beta^2+\beta^2G(\beta^2)(3c_2+4c_1\beta^2)]\mathscr{D}_1\phi\mathscr{D}_1^4\phi$$
$$+[1-4c_0\beta^2-8\beta^2G(\beta^2)(\tfrac{3}{2}c_2-c_1\beta^2)]\mathscr{D}_1^2\phi\mathscr{D}_1^3\phi)](x,\theta)\,. \tag{30}$$

The corresponding conserved charge Q_3 generates a non-linear automorphism on the local field algebra [see the contact terms in (22)]:

$$[Q_3,\phi(x,\theta)]=2\mathscr{D}_1^6\phi(x,\theta)+4\beta^2[1-3\beta^2(c_2+2c_3)+6c_2\beta^4]^{-1}$$
$$\times(\mathrm{N}[(\mathscr{D}_1^2\phi)^3](x,\theta)-\tfrac{3}{2}\mathrm{N}[\mathscr{D}_1\phi\mathscr{D}_1^2\phi\mathscr{D}_1^3\phi](x,\theta))\,. \tag{31}$$

3. Critical point in the SSG model

Let us consider the quantum conserved spin-vector supercurrent of the SSG model. Classically the latter has the form:

$$J^1_{3/2\mathrm{cl}}(x,\theta)=\mathscr{D}_1\phi\mathscr{D}_1^2\phi,\qquad J^2_{3/2\mathrm{cl}}(x,\theta)=-\frac{m}{\beta^2}\mathscr{D}_1(\cos\beta\phi)\,.$$

On quantum level the corresponding WI must be satisfied:

$$\mathscr{D}_2\langle J^1_{3/2}(x,\theta)X_\phi\rangle-\mathscr{D}_1\langle J^2_{3/2}(x,\theta)X_\phi\rangle=-i\sum_{l=1}^{L_\phi}[\delta(x-x_l)\,\delta(\theta-\theta_l)]$$
$$\times\langle\mathscr{D}_1^2\phi(x,\theta)\hat{X}^l_\phi\rangle+i\sum_{l=1}^{L_\phi}\mathscr{D}_1[\delta(x-x_l)\,\delta(\theta-\theta_l)]\langle\mathscr{D}_1\phi(x,\theta)\hat{X}^l_\phi\rangle\,. \tag{32}$$

Substituting $J^1_{3/2}(x,\theta)=\mathrm{N}[\mathscr{D}_1\phi\mathscr{D}_1^2\phi](x,\theta)$ into the l.h.s. of (32) and using QEM (18) and the ZI for the arising "anomalous" term (fig. 3),

$$m^2\langle\mathrm{N}[\mathscr{D}_1^2\phi\{(s^2-1)\phi\}+\mathscr{D}_1\phi\mathscr{D}_1\{(s^2-1)\phi\}](x,\theta)X_\phi\rangle$$
$$=\frac{\tilde{m}}{4\pi}\langle\mathrm{N}[\mathscr{D}_1(\cos\beta\phi)](x,\theta)X_\phi\rangle\,, \tag{33}$$

we obtain for the quantum "improved" spin-vector supercurrent the expression:

$$J^1_{3/2}(x,\theta)=\mathrm{N}[\mathscr{D}_1\phi\mathscr{D}_1^2\phi](x,\theta)\,,$$
$$J^2_{3/2}(x,\theta)=-\left(1-\frac{\beta^2}{4\pi}\right)\frac{\tilde{m}}{\beta^2}\mathrm{N}[\mathscr{D}_1(\cos\beta\phi)](x,\theta)\,. \tag{34}$$

Introducing in (34) the expansion of $\phi(x,\theta)$ in components (1) we find the "improved" quantum energy-momentum tensor (in light cone components):

$$\theta_{\xi\xi}(x)=\mathrm{N}[(\partial_\xi\varphi)^2-i\psi_2\partial_\xi\psi_2](x)\,,$$
$$\theta_{\eta\eta}(x)=\mathrm{N}[(\partial_\eta\varphi)^2+i\psi_1\partial_\eta\psi_1](x)\,,$$
$$\theta_{\xi\eta}(x)=\theta_{\eta\xi}(x)=\theta^\mu_\mu(x)$$
$$=\left(1-\frac{\beta^2}{4\pi}\right)\tilde{m}\mathrm{N}\left[\frac{iF}{\beta}\sin\beta\varphi-\psi_1\psi_2\cos\beta\varphi\right](x)\,. \tag{35}$$

Similarly one can consider the dilatation WIs (i.e., the Callan–Symanzik equations [32]) in the SSG model. Applying the standard method of differential-vertex operations [33] on the effective lagrangian [29] and using (33) we derive the following homogeneous equations:

$$(\mu\,\partial/\partial\mu+R(\beta^2,m/\mu)m\,\partial/\partial m)\langle X_\phi\rangle=0\,,$$
$$R(\beta^2,m/\mu)\equiv\left[1-(1-\beta^2/4\pi)\frac{\tilde{m}}{\mu\,\partial\tilde{m}/\partial\mu}\right]^{-1}; \tag{36}$$

$$[\mu\partial/\partial\mu + R(\beta^2, m/\mu)m\partial/\partial m + R(\beta^2, m/\mu)\beta^2/4\pi]\langle \mathrm{N}[\cos\beta\phi](x,\theta)X_\phi\rangle = 0\,. \tag{37}$$

Eq. (36) [or equivalently eq. (35)] shows directly that $\beta^2 = 4\pi$ is a point of scale invariance. From (37) we read off the total anomalous UV dimension of the composite operator $\mathrm{N}[\cos\beta\phi](x,\theta)$:

$$\dim(\mathrm{N}[\cos\beta\phi]) = \frac{\beta^2}{4\pi}R(\beta^2, m/\mu)\,, \qquad (=1 \text{ for } \beta^2 = 4\pi)\,. \tag{38}$$

Formulas (35)–(37) may be compared with their counterparts in the usual SG model [34]:

$$\mathcal{L}^{\mathrm{eff}}_{\mathrm{SG}}(x) = \tfrac{1}{2}\mathrm{N}_2[(\partial_\mu u)^2 - m^2u^2](x) + \mathrm{N}_0\Big[\tfrac{1}{2}m^2u^2 + \frac{\tilde{m}}{\beta^2_{\mathrm{SG}}}(\cos\beta_{\mathrm{SG}}u - 1)\Big](x):$$

$$\theta^{\mathrm{SG}\mu}_{\mu} = \theta^{\mathrm{SG}}_{\xi\eta} = \theta^{\mathrm{SG}}_{\eta\xi} = 2(1-\beta^2_{\mathrm{SG}}/8\pi)\tilde{m}^2/\beta^2_{\mathrm{SG}}(\mathrm{N}[\cos\beta_{\mathrm{SG}}u](x) - 1)\,;$$

$$[\mu\partial/\partial\mu + r(\beta^2_{\mathrm{SG}}, m^2/\mu^2)(m\partial/\partial m + \beta^2_{\mathrm{SG}}/4\pi)]\langle \mathrm{N}[\cos\beta_{\mathrm{SG}}u](x)X_u\rangle = 0\,,$$

$$r(\beta^2_{\mathrm{SG}}, m^2/\mu^2) = [1-(1-\beta^2_{\mathrm{SG}}/8\pi)\tilde{m}^2(\mu^2\partial\tilde{m}^2/\partial\mu^2)^{-1}]^{-1}\,. \tag{39}$$

It should be noted that eqs. (39) (derived in the BPHZ framework) are apparently true in the region $0<\beta^2_{\mathrm{SG}}<4\pi$ as it follows from the rigorous results of ref. [35]. On the other hand, the exact treatment of the SG model based on the quantum inverse scattering method (second of refs. [14]) shows that there is no coupling constant renormalization in the whole region $0 \leqslant \beta^2_{\mathrm{SG}} \leqslant 8\pi$. Consequently, eqs. (39) also remain valid for $4\pi \leqslant \beta^2_{\mathrm{SG}} \leqslant 8\pi$ with a possible change of the β^2_{SG} dependence of the corresponding coefficient functions due to the arising non-leading short-distance singularities [24]. This change depends on the particular choice of a modified renormalization presciption for the composite operator $\mathrm{N}[\cos\beta_{\mathrm{SG}}u](x)$. However, the facts that $\beta^2_{\mathrm{SG}} = 8\pi$ is a critical point and that at this point $\dim(\mathrm{N}[\cos\beta_{\mathrm{SG}}u]) = 2$ (strict renormalizability) which follow from eqs. (39) are indeed true, as it was rigorously proved in the second of refs. [14].

It is not clear to us whether there are non-leading short-distance singularities also in the SSG model for β^2 in the vicinity of the critical point 4π (which would eventually require a modification of the renormalization procedure). Nevertheless the parallelism with the usual SG model indicates that the conclusions based on eqs. (36)–(38) about the critical character of the point $\beta^2 = 4\pi$ are similarly trustworthy.

The critical point $\beta^2_{\mathrm{SG}} = 8\pi$ is identified in ref. [23] as the anti-ferromagnetic phase-transition point of the *XYZ* model. The analogous interpretation of the SSG critical point $\beta^2 = 4\pi$ is at present not known.

4. Infinite series of HLQCCs in the SSG model

To construct the whole infinite series of HLQCCs $J^{\alpha}_{2k+3/2}(x,\theta)$ $(k \geqslant 2)$ according to the general algebraic normal product scheme [16, 20, 19] which would lead to conservation laws generalizing (20):

$$\sum_{\text{in}} (p_{l,\eta})^{\pm(2k+1)} = \sum_{\text{out}} (p_{l,\eta})^{\pm(2k+1)}, \tag{20'}$$

we note that $J^1_{2k+3/2}(x,\theta)$ should have the following general form:

$$J^1_{2k+3/2}(x,\theta) = \sum_{R=1}^{k+1} \beta^{2(R-1)} \sum_{\{(r)\}} \alpha^{(k)}_{\{(r)\}}(\beta^2) \mathrm{N}[Y^{(k)}_{\{(r)\}}(\phi)](x,\theta), \tag{40}$$

$$Y^{(k)}_{\{(r)\}}(\phi) \equiv \mathscr{D}_1^{r_1}\phi \ldots \mathscr{D}_1^{r_{2R}}\phi,$$

$$\{(r)\} \equiv \{r_1, \ldots, r_{2R}\}, \qquad 1 \leqslant r_1 \leqslant \ldots \leqslant r_{2R}, \qquad \sum_{c=1}^{2k} r_c = 2(2k+\tfrac{3}{2}).$$

The second sum on the r.h.s. of (40) runs over the set of all composite operators $Y^{(k)}_{\{(r)\}}(\phi)$ which are linearly independent up to total $\mathscr{D}_1$ derivatives, with Lorentz weights and UV dimensions: $w(Y^{(k)}_{\{(r)\}}) = \dim(Y^{(k)}_{\{(r)\}}) = 2k+\frac{3}{2}$. Let $a(k)$ denote their number. All coefficients $\alpha^{(k)}_{\{(r)\}}$ (power series in β^2) are to be determined from the WIs:

$$\begin{aligned}
&\mathscr{D}_2\langle J^1_{2k+3/2}(x,\theta)X_\phi\rangle = \mathscr{D}_1\langle J^2_{2k+3/2}(x,\theta)X_\phi\rangle \\
&\quad + \sum_{A=1}^{k} \beta^{2A} \sum_{\{(i)\}} \left[\sum_{\{(r)\}} h^{(k)}_{\{(i)\}\{(r)\}}\, \alpha^{(k)}_{\{(r)\}}\right] \\
&\quad \times \left\langle \mathrm{N}\left[Y^{(k)}_{\{(i)\}} \frac{\tilde{m}}{\beta} \sin\beta\phi\right](x,\theta)X_\phi\right\rangle \\
&\quad + \sum_{B=1}^{k} \beta^{2(B-1)} \sum_{\{(j)\}} [h^{(k)}_{\{(j)\}\{(r)\}}\alpha^{(k)}_{\{(r)\}}] \\
&\quad \times \langle \mathrm{N}[Y^{(k)}_{\{(j)\}}\tilde{m}\cos\beta\phi](x,\theta)X_\phi\rangle \\
&\quad + i\sum_{R=1}^{k+1} \beta^{2(R-1)} \sum_{\{(r)\}} \alpha^{(k)}_{\{(r)\}} \sum_{c=1}^{2R} (-1)^{r_c(1+\sum_{b=1}^{c-1} r_b)} \\
&\quad \times \sum_{l=1}^{L_\phi} [\mathscr{D}_1^{r_c-1}\,\delta(x-x_l)\,\delta(\theta-\theta_l)] \\
&\quad \times \langle \mathrm{N}[\mathscr{D}_1^{r_1}\phi \ldots \hat{c} \ldots \mathscr{D}_1^{r_{2R}}\phi](x,\theta)\hat{X}^l_\phi\rangle;
\end{aligned} \tag{41}$$

$$\{(i)\} \equiv \{i_1, \ldots, i_{2A+1}\}, \qquad 1 \leqslant i_1 \leqslant \ldots \leqslant i_{2A+1}, \qquad \sum_{a=1}^{2A+1} i_a = 2(2k+1),$$

$$\{(j)\} \equiv \{j_1, \ldots, j_{2B}\}, \qquad 1 \leqslant j_1 \leqslant \ldots \leqslant j_{2B}, \qquad \sum_{b=1}^{2B} j_b = 2(2k+1); \tag{42}$$

$$J^2_{2k+3/2}(x,\theta)=\sum_{A'=1}^{k}\beta^{2A'}\sum_{\{(\rho)\}}b^{(k)}_{\{(\rho)\}}(\beta^2)\Big\langle \mathrm{N}\Big[Y^{(k)}_{\{(\rho)\}}\frac{\tilde{m}}{\beta}\sin\beta\phi\Big](x,\theta)X_\phi\Big\rangle$$

$$+\sum_{B'=2}^{k}\beta^{2(B'-1)}\sum_{\{(\sigma)\}}b^{(k)}_{\{(\sigma)\}}(\beta^2)\langle \mathrm{N}[Y^{(k)}_{\{(\sigma)\}}\tilde{m}\cos\beta\phi](x,\theta)X_\phi\rangle\,,$$

$$\{(\rho)\}\equiv\{\rho_1,\dots,\rho_{2A'+1}\},\qquad 1\leqslant\rho_1\leqslant\dots\leqslant\rho_{2A'+1}\,,$$

$$\sum_{a=1}^{2A'+1}\rho_a=2(2k+\tfrac{1}{2})\,,$$

$$\{(\sigma)\}\equiv\{\sigma_1,\dots,\sigma_{2B'}\},\qquad 1\leqslant\sigma_1\leqslant\dots\leqslant\sigma_{2B'}\,,$$

$$\sum_{b=1}^{2B'}\sigma_b=2(2k+\tfrac{1}{2})\,, \tag{43}$$

by the requirement that the current $J^\alpha_{2k+3/2}(x,\theta)$ is conserved. That is, $\alpha^{(k)}_{\{(r)\}}$ should satisfy the following linear algebraic system of $c(k)$ equations [$c(k)$ is the number of allowed configurations $\{(i)\}$, $\{(j)\}$, see below]:

$$H^{(k)}_{\{(\cdot)\}}(\beta^2)\equiv\sum_{\{(r)\}}{}' h^{(k)}_{\{(\cdot)\}\{(r)\}}(\beta^2)\alpha^{(k)}_{\{(r)\}}(\beta^2)+h^{(k)}_{\{(\cdot)\}\{2k+1,\,2k+2\}}(\beta^2)=0\,,$$

$$\{(\cdot)\}=\{(i)\},\{(j)\}\,. \tag{44}$$

The sums in the non-total-divergence terms on the r.h.s. of (41) run over all possible sets of composite operators $Y^{(k)}_{\{(i)\}}(\phi)$, $Y^{(k)}_{\{(j)\}}(\phi)$ which are linearly independent up to total $\mathscr{D}_1$ derivatives, with Lorentz weights and UV dimensions indicated in (42). The coefficient functions $h^{(k)}_{\{(\cdot)\}\{(r)\}}(\beta^2)$ get contributions from two sources: the "classical" part of QEM (18) applied to the l.h.s. of (41) and the ZIs for the corresponding "anomalous" anisotropic normal products (see fig. 4). The particularly simple structure of the latter is due to the superrenormalizability (11′) and to the peculiarity of two-dimensional Lorentz kinematics as already stated in subsect. 2.4. In eqs. (44) we have distinguished the terms corresponding to $\{(r)\}=\{2k+1,2k+2\}$ and we have set $\alpha^{(k)}_{\{2k+1,2k+2\}}=1$ (*cf.* (21) where $F=1$).

Elementary calculations give $a(k)=c(k)$ only for $k=0,1$ (i.e., for the already constructed $J^\alpha_{3/2}(x,\theta)$, $J^\alpha_{7/2}(x,\theta)$) and for $k=2$ (in the case of the usual SG model $a(k)=c(k)$ for $j^{(2k+1)}_\mu(x)$, $k=0,1,2,3$ [16, 34]). The $k=2$ HLQCC (40) has the following form:

$$J^1_{11/2}(x,\theta)=\mathrm{N}[\mathscr{D}_1^5\phi\mathscr{D}_1^6\phi+\beta^2\alpha_1(\beta^2)\mathscr{D}_1\phi\mathscr{D}_1^2\phi(\mathscr{D}_1^4\phi)^2$$

$$+\beta^2\alpha_2(\beta^2)(\mathscr{D}_1^2\phi)^2\mathscr{D}_1^3\phi\mathscr{D}_1^4\phi+\beta^4\alpha_3(\beta^2)\mathscr{D}_1\phi(\mathscr{D}_1^2\phi)^5](x,\theta)\,, \tag{45}$$

where all coefficients $\alpha_i(\beta^2)$, $i=1,2,3$, are directly computable in terms of supergraphs of the type depicted in fig. 4 (the zeroth orders of $\alpha_i(\beta^2)$ coincide with the coefficients in the corresponding classical expression for $J^1_{11/2}(x,\theta)$).

Unfortunately $c(k) > a(k)$ for $k \geq 3$, i.e., the system (44) becomes overdetermined and the question arises as to whether (44) is compatible. To this end we note that up to now we have not used any information from the existence of non-trivial higher non-linear symmetries generated by the already constructed HLQCCs (30), (45) [see eq. (31)]. However, because of the non-linearity it is too hard to extract directly any useful consequences and it is preferable to go over to asymptotic fields in order to linearize the action of the corresponding charges Q_3 and Q_5.

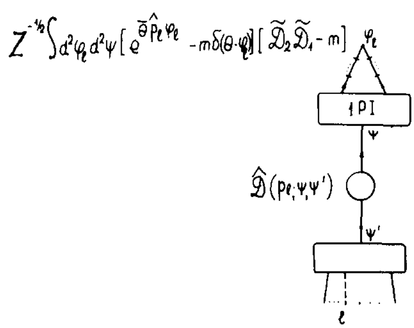

Fig. 5. Amputation and going on-mass-shell for $\langle \mathrm{N}[\mathscr{D}_1^{r_c-1}(\mathscr{D}_1^{r_1}\phi \dots \hat{c} \dots \mathscr{D}_1^{r_{2R}}\phi)](p_l, \varphi_l)\tilde{X}^l_\Phi\rangle$. The upper block subgraph represents $\langle \mathrm{N}[\mathscr{D}_1^{r_c-1}(\mathscr{D}_1^{r_1}\phi \dots \hat{c} \dots \mathscr{D}_1^{r_{2R}}\phi)](p_l, \varphi_l)\tilde{\phi}(-p_l, \psi)\rangle^{1PI}$. The total propagator has the general supersymmetric form: $\hat{\mathscr{D}}(p; \psi, \psi') = [(1+\zeta(p^2/m^2)) \times (m^2 - p^2 - i0)]^{-1}(\mathrm{e}^{\bar{\psi}\hat{p}\psi'} - m\delta(\psi - \psi'))$; $Z^{-1} \equiv 1 + \zeta(1)$.

It is well-known [36] that the asymptotic limit in terms of Green functions in momentum space means amputation of external propagators plus going on-mass-shell. Application of these operations on integrated (over x, θ) contact terms in the WIs (22), (41) gives the result (see fig. 5):

$$
\begin{aligned}
&\langle \mathcal{N}[\mathscr{D}_1^{r_c-1}(\mathscr{D}_1^{r_1}\phi \dots \hat{c} \dots \mathscr{D}_1^{r_{2R}}\phi)](p_l, \theta_l)\tilde{X}^l_\Phi\rangle^{\mathrm{Amp}}\big|_{\mathrm{m.sh.}} \\
&\quad = (p_{l,\eta})^{2k+1} m (B(m^2) - A(m^2))\langle \mathrm{out}|\mathrm{in}\rangle\,; \\
&\langle \mathrm{out}|\mathrm{in}\rangle \equiv {}_{\mathrm{out}}\langle p_1, \varphi_1; \dots ; p_{L_{\mathrm{out}}}, \varphi_{L_{\mathrm{out}}}| \dots ; p_{L_\Phi}, \varphi_{L_\Phi}\rangle_{\mathrm{in}} \\
&\quad = \left\{ \int \prod_{l=1}^{L_\Phi} d^2\theta_l \prod_{\mathrm{in}} [\mathrm{e}^{\bar{\varphi}_l \hat{p}_l \theta_l} - m\delta(\varphi_l - \theta_l)] \langle X_\Phi\rangle^{\mathrm{Amp}} \prod_{\mathrm{out}} [\mathrm{e}^{\bar{\theta}_{l'}\hat{p}_{l'}\varphi_{l'}} \right. \\
&\qquad \left. - m\delta(\theta_{l'} - \varphi_{l'})] \right\}_{\mathrm{m.sh.}}\,, \qquad (46)
\end{aligned}
$$

where the following general supersymmetric representation was used:

$$
\begin{aligned}
&\langle \mathrm{N}[\mathscr{D}_1^{r_c-1}(\mathscr{D}_1^{r_1}\phi \dots \hat{c} \dots \mathscr{D}_1^{r_{2R}}\phi)](p, \theta)\tilde{\phi}(-p, \varphi)\rangle^{1\mathrm{PI}} \\
&\quad = p_\eta^{2k+1}[A(p^2)\,\mathrm{e}^{\bar{\theta}\hat{p}\varphi} + mB(p^2)\delta(\theta - \varphi)]\,,
\end{aligned}
$$

$A(p^2)$, $B(p^2)$ being Lorentz invariant functions (1PI ≡ one-particle irreducible).

Hereby the integrated WIs (41) reduce to the form:

$$\sum_{A=1}^{k} \beta^{2A} \sum_{\{(i)\}} H^{(k)}_{\{(i)\}}(\beta^2)\langle\text{out}| \int d^2x\, d^2\theta\, \mathrm{N}\left[Y^{(k)}_{\{(i)\}} \frac{\tilde{m}}{\beta} \sin \beta\phi \right](x, \theta)|\text{in}\rangle$$

$$+ \sum_{B=1}^{k} \beta^{2(B-1)} \sum_{\{(j)\}} H^{(k)}_{\{(j)\}}(\beta^2)\langle\text{out}| \int d^2x\, d^2\theta\, \mathrm{N}[Y^{(k)}_{\{(j)\}} \tilde{m} \cos \beta\phi](x, \theta)|\text{in}\rangle$$

$$= \text{const}(\beta^2) \sum_{l=1}^{L_\phi} p^{2k+1}_{l,\eta} \langle\text{out}|\text{in}\rangle . \tag{47}$$

If (44) is compatible ($H^{(k)}_{\{(\cdot)\}} = 0$), eqs. (47) give the conservation laws (20′) and also:

$$\langle\text{out}|\text{in}\rangle \Big|_{\sum_{l=1}^{L_\phi} p^{2k+1}_{l,\eta} \neq 0} = 0. \tag{48}$$

According to the above results, eq. (48) is certainly valid for $k = 1, 2$. Now we are ready to employ a supersymmetric generalization of the arguments of ref. [19] in order to show the compatibility of (44) for all k*.

Let us assume for the time being that (44) may not be satisfied. Then we can pick up from (44) a quadratic subsystem of $a(k)$ equations, solve it for $\alpha^{(k)}_{\{(r)\}}(\beta^2)$ (the corresponding determinant is of the form const$+\mathrm{O}(\beta^2)$) and substitute the result into the remaining $c(k)-a(k)$ $H^{(k)}_{\{(\cdot)\}'}(\beta^2)$ (which may not be zero). On account of (48) for $k = 1$ we have:

$$[\text{l.h.s. of } (47)]' \Big|_{\sum_{L=1}^{L_\phi} p^{3}_{l,\eta} \neq 0} = 0 . \tag{49}$$

Here the prime indicates that only terms with $H^{(k)}_{\{(\cdot)\}'}(\beta^2)$ eventually survive. We shall see below that, actually, eqs. (49) are satisfied without the constraint on the external momenta and, moreover, they imply [*cf.* (40), (43)]

$$\sum_{A=1}^{k} \beta^{2A} \sum_{\{(i)\}'} H^{(k)}_{\{(i)\}'}(\beta^2)\left\langle \mathrm{N}\left[Y^{(k)}_{\{(i)\}'} \frac{\tilde{m}}{\beta} \sin \beta\phi \right](x, \theta) X_\phi \right\rangle$$

$$+ \sum_{B=1}^{k} \beta^{2(B-1)} \sum_{\{(j)\}'} H^{(k)}_{\{(j)\}'}(\beta^2)\langle \mathrm{N}[Y^{(k)}_{\{(j)\}'} \tilde{m} \cos \beta\phi](x, \theta) X_\phi \rangle$$

$$= \mathscr{D}_2 \left\{ \sum_{R=1}^{k+1} \beta^{2(R-1)} \sum_{\{(r)\}} \hat{\alpha}^{(k)}_{\{(r)\}}(\beta^2, H')\langle \mathrm{N}[Y^{(k)}_{\{(r)\}}](x, \theta) X_\phi \rangle \right\}$$

$$+ \mathscr{D}_1 \left\{ \sum_{A'=1}^{k} \beta^{2A'} \sum_{\{(\rho)\}} \hat{b}^{(k)}_{\{(\rho)\}}(\beta^2, H')\left\langle \mathrm{N}\left[Y^{(k)}_{\{(\rho)\}} \frac{\tilde{m}}{\beta} \sin \beta\phi \right](x, \theta) X_\phi \right\rangle \right.$$

$$\left. + \sum_{B'=2}^{k} \beta^{2(B'-1)} \sum_{\{(\sigma)\}} \hat{b}^{(k)}_{\{(\sigma)\}}(\beta^2, H')\langle \mathrm{N}[Y^{(k)}_{\{(\sigma)\}} \tilde{m} \cos \beta\phi](x, \theta) X_\phi \rangle \right\} , \tag{50}$$

* Actually ref. [19] gives a rigorous proof for the case of the quantum SG and massive Thirring models of a "folklore" statement [L. D. Faddeev, private communication (1976)] that in every completely integrable model the existence of a first HLQCC implies the existence of an infinite number of HLQCCs.

where all coefficient functions $\hat{\alpha}^{(k)}_{\{(r)\}}$, $\hat{b}^{(k)}_{\{(\cdot)\}}$ depend linearly on the coefficients of the β^2 power-series expansion of $H^{(k)}_{\{(\cdot)\}'}(\beta^2)$. From (50) we would deduce that the WI (41) is satisfied for a modified current $\hat{J}^{\alpha}_{2k+3/2}(x,\theta)$, (40), (43), with $\alpha^{(k)}_{\{(r)\}}$, $b^{(k)}_{\{(\cdot)\}}$ replaced by

$$\alpha^{(k)}_{\{(r)\}}(\beta^2)+\hat{\alpha}^{(k)}_{\{(r)\}}(\beta^2,H'), \qquad b^{(k)}_{\{(\cdot)\}}(\beta^2)+\hat{b}^{(k)}_{\{(\cdot)\}}(\beta^2,H'),$$

respectively. This should mean that $\alpha^{(k)}_{\{(r)\}}(\beta^2)+\hat{\alpha}^{(k)}_{\{(r)\}}(\beta^2,H')$ already constitute a solution to the whole overdetermined system (44) what contradicts the assumption of the incompatibility of the latter and the fact that $\alpha^{(k)}_{\{(r)\}}(\beta^2)$ are, by construction, a solution to the corresponding quadratic subsystem. Therefore $H^{(k)}_{\{(\cdot)\}'}(\beta^2)=0$ [and consequently $\hat{\alpha}^{(k)}_{\{(r)\}}(\beta^2,H')=0$], eq. (44) is compatible and hereby the construction of the corresponding current (40) is completed.

The proof of (50) goes recursively order by order in β^2 perturbation theory. It is sufficient to verify (50) to lowest order. All further steps follow the pattern of ref. [19] and need not be reproduced here.

The lowest (zeroth) orders of the matrix elements in (47) [and (49)] arise when $L_\phi=2A+2$ or $L_\phi=2B$ respectively [in the notations of (42)]. They are of the form (without integration over x)

$$\begin{aligned}
&\langle\text{out}|\int \mathrm{d}^2\theta\,\mathrm{N}\Big[Y^{(k)}_{\{(\cdot)\}}\Big(\begin{matrix}\tilde{m}\cos\beta\phi\\(\tilde{m}/\beta)\sin\beta\phi\end{matrix}\Big)\Big](0,\theta)|\text{in}\rangle^{(0)}=m\prod_{l=1}^{L_\phi}(1-m\,\delta(\varphi_l))\\
&\times\sum_{s=0}^{N(k)}\sum_{a_1,\dots,a_{2s}=1}^{L_\phi}\Big\{\frac{1}{2s(2s-1)}\Big[-P^{\{(\cdot)\}}_{a_1\dots a_{2s-2}}\frac{1}{p_{a_{2s-1}}}+\sum_{j=1}^{2s-2}P^{\{(\cdot)\}}_{\underset{(j)}{a_1\dots a_{2s-1}\dots a_{2s-2}}}\frac{1}{p_{a_j}}\\
&-\sum_{j=1}^{2s-1}(a_j\to a_{2s})\Big]-L_\phi P^{\{(\cdot)\}}_{a_1\dots a_{2s}}+\sum_{l=1}^{L_\phi}\Big[\sum_{j=1}^{2s}P^{\{(\cdot)\}}_{\underset{(j)}{a_1\dots l\dots a_{2s}}}\Big(1+\frac{p_l}{p_{a_j}}\Big)\Big]\\
&+(2s+1)(s+1)\sum_{l_1=1}^{L_\phi}\sum_{l_2=1}^{L_\phi}(p_{l_1}-p_{l_2})P^{\{(\cdot)\}}_{a_1\dots a_{2s}l_1l_2}\Big\}\prod_{h=1}^{2s}(\varphi^2_{a_h}p_{a_h}+m\varphi^1_{a_h})\,;\\
&P^{\{(\cdot)\}}_{a_1\dots a_{2M}}\equiv 0,\qquad M\geqslant N(k)\,.
\end{aligned}\tag{51}$$

Here $P^{\{(\cdot)\}}_{a_1\dots a_{2s}}$ denote homogeneous polynomials in $p_l\equiv p_{l,\eta}$ (the light-cone index η is suppressed for brevity) of degree $2k-s$, totally antisymmetric under permutations of the indices $a_1,\dots,a_{2s}$ (with the momenta fixed) and symmetric under permutations of $p_{b_1},\dots,p_{b_{L_\phi-2s}}$ where $\{b_1,\dots,b_{L_\phi-2s}\}$ is the complementary subset of $\{a_1,\dots,a_{2s}\}$ to $\{1,\dots,L_\phi\}$. $2(N(k)-1)$ is the maximal number of odd integers i_a or j_b belonging to the corresponding sets $\{(i)\}$, $\{(j)\}$, (42), (with fixed $2A+2=2B=L_\phi$) appearing on the l.h.s. of (51). To lowest order eq. (49), on account of (51), implies:

$$P_{a_1\dots a_{2s}} \equiv \sum_{\{(\cdot)\}',2A+2=2B=L_\phi} [H^{(k)}_{\{(\cdot)\}'}]^{(0)} P^{\{(\cdot)\}'}_{a_1\dots a_{2s}}\Bigg|_{\substack{\sum_{l=1}^{L_\phi} p_l = \sum_{l=1}^{L_\phi} (p_l)^{-1}=0,\\ \sum_{l=1}^{L_\phi} p_l^3 \neq 0}} = 0\,. \tag{52}$$

Applying a general theorem on the zeros of polynomials in several (complex) variables (see appendix A of ref. [19]) to $P_{a_1\dots a_{2s}}$, (52), we get for the latter the following representation:

$$P_{a_1\dots a_{2s}} = \Big(\sum_{l=1}^{L_\phi} p_l\Big) Q^{(+)}_{a_1\dots a_{2s}} + \Big(\sum_{l=1}^{L_\phi} p_l^{-1}\Big) Q^{(-)}_{a_1\dots a_{2s}}\,, \tag{53}$$

where $Q^{(\pm)}_{a_1\dots a_{2s}}$ are homogeneous polynomials in p_l of degree $2k-s\mp1$, respectively, all other symmetry properties being the same as for $P_{a_1\dots a_{2s}}$. Moreover $Q^{(-)}_{a_1\dots a_{2s}}$ contain a factor $(\prod_l p_l)$, $l=1,\dots,L_\phi$. To prove (50) to lowest order we note that the following general representations are valid [in the notations of (40), (43)]:

$$\begin{aligned}
\langle\mathrm{out}|\int \mathrm{d}^2\theta\, \mathscr{D}_2 N[Y^{(k)}_{\{(r)\}}](0,\theta)|\mathrm{in}\rangle^{(0)} &= m\Big(\sum_{l=1}^{L_\phi} p_l^{-1}\Big)\prod_{l=1}^{L_\phi}(1-m\delta(\varphi_l))\\
\times \sum_{s=0}^{N(k)} \sum_{a_1,\dots,a_{2s}=1}^{L_\phi} \Big\{\frac{1}{2s}\Big[-P'^{\{(r)\}}_{a_1\dots a_{2s-1}} &+ \sum_{j=1}^{2s-1} P'^{\{(r)\}}_{a_1\dots \underset{(j)}{a_{2s}} \dots a_{2s-1}}\Big]\\
-(2s+1)\sum_{l=1}^{L_\phi} p_l P'^{\{(r)\}}_{a_1\dots a_{2s} l}\Big\} &\prod_{h=1}^{2s}(\varphi^2_{a_h} p_{a_h} + m\varphi^1_{a_h}), \qquad 2R=L_\phi\,;
\end{aligned} \tag{54a}$$

$$\begin{aligned}
\langle\mathrm{out}|\int \mathrm{d}^2\theta\, \mathscr{D}_1 N\Big[Y^{(k)}_{\{(\cdot)\}}\begin{pmatrix}\tilde m\cos\beta\phi\\ \frac{\tilde m}{\beta}\sin\beta\phi\end{pmatrix}\Big](0,\theta)|\mathrm{in}\rangle^{(0)} &= m\Big(\sum_{l=1}^{L_\phi} p_l\Big)\prod_{l=1}^{L_\phi}(1-m\delta(\varphi_l))\\
\times \sum_{s=0}^{N(k)} \sum_{a_1,\dots,a_{2s}=1}^{L_\phi} \Big\{\frac{1}{2s}\Big[-P''^{\{(\cdot)\}}_{a_1\dots a_{2s-1}}\frac{1}{p_{a_{2s}}} &+ \sum_{j=1}^{2s-1} P''^{\{(\cdot)\}}_{a_1\dots \underset{(j)}{a_{2s}} \dots a_{2s-1}}\frac{1}{p_{aj}}\Big]\\
-(2s+1)\sum_{l=1}^{L_\phi} P''^{\{(\cdot)\}}_{a_1\dots a_{2s} l}\Big\} &\prod_{h=1}^{2s}(\varphi^2_{a_h} p_{a_h} + m\varphi^1_{a_h})\,;\\
\{(\cdot)\} = \{(\sigma)\},\{(\rho)\}, \qquad 2A'+2=2B'=L_\phi\,.&
\end{aligned} \tag{54b}$$

Here $P'^{,''\,\{(\cdot)\}}_{a_1\dots a_{2s+1}}$ are homogeneous polynomials in p_l with the same properties as $Q^{(\mp)}_{a_1\dots a_{2s}}$. Clearly, the correspondence between the expressions on the right-hand sides of (51), (54a, b) and the composite operators on the left-hand sides is one to one. Therefore we take matrix elements (to lowest order) of both sides of eqs. (50).

The resulting equations, on account of eqs. (51), (53), (54a, b) and of the linear independence of all Grassmann factors $\prod_h (\varphi^2_{a_h} p_{a_h} + m\varphi^1_{a_h})$, reduce to a set of algebraic equations for a determination of the unknown zeroth-order coefficients $[\hat{a}^{(k)}_{\{(r)\}}]^{(0)}$, $[\hat{b}^{(k)}_{\{(\cdot)\}}]^{(0)}$ in terms of $Q^{(\pm)}_{a_1 \ldots a_{2s}}$:

$$\frac{1}{2s}\Big[-P'_{a_1\ldots a_{2s-1}} + \sum_{j=1}^{2s-1} P'_{a_1\ldots \underset{(j)}{a_{2s}} \ldots a_{2s-1}}\Big] - (2s+1)\sum_{l=1}^{L_\Phi} p_l P'_{a_1\ldots a_{2s} l}$$

$$= \frac{1}{2s(2s-1)}\Big[-Q^{(-)}_{a_1\ldots a_{2s-2}}\frac{1}{p_{a_{2s-1}}} + \sum_{j=1}^{2s-2} Q^{(-)}_{a_1\ldots \underset{(j)}{a_{2s-1}} \ldots a_{2s-2}}\frac{1}{p_{a_j}}$$

$$- \sum_{j=1}^{2s-1}(a_j \to a_{2s})\Big] - L_\Phi Q^{(-)}_{a_1\ldots a_{2s}} + \sum_{l=1}^{L_\Phi}\Big[\sum_{j=1}^{2s} Q^{(-)}_{a_1\ldots \underset{(j)}{l}\ldots a_{2s}}\Big(1+\frac{p_l}{p_{a_j}}\Big)\Big]$$

$$+ (2s+1)(s+1)\sum_{l_1=1}^{L_\Phi}\sum_{l_2=1}^{L_\Phi}(p_{l_1}-p_{l_2}) Q^{(-)}_{a_1\ldots a_{2s} l_1 l_2}; \tag{55a}$$

$$\frac{1}{2s}\Big[-P''_{a_1\ldots a_{2s-1}}\frac{1}{p_{a_{2s}}} + \sum_{j=1}^{2s-1} P''_{a_1\ldots \underset{(j)}{a_{2s}} \ldots a_{2s-1}}\frac{1}{p_{a_j}}\Big] - (2s+1)\sum_{l=1}^{L_\Phi} P''_{a_1\ldots a_{2s} l}$$

$$= [\text{the same r.h.s. as in (55a) with } Q^{(-)} \to Q^{(+)}]; \tag{55b}$$

$$P'_{a_1\ldots a_{2s+1}} \equiv \sum_{\{(r)\}, 2R = L_\Phi} [\hat{a}^{(k)}_{\{(r)\}}]^{(0)} P'^{\{(r)\}}_{a_1\ldots a_{2s+1}},$$

$$P''_{a_1\ldots a_{2s+1}} \equiv \sum_{\{(\rho)\}, 2A'+2 = L_\Phi} [\hat{b}^{(k)}_{\{(\rho)\}}]^{(0)} P''^{\{(\rho)\}}_{a_1\ldots a_{2s+1}} + \sum_{\{(\sigma)\}, 2B' = L_\Phi} [\hat{b}^{(k)}_{\{(\sigma)\}}]^{(0)} P''^{\{(\sigma)\}}_{a_1\ldots a_{2s+1}}.$$

Straightforward (although somewhat lengthy) calculations give the following solutions of the (overdetermined!) systems (55a, b):

$$P'_{a_1\ldots a_{2s+1}} = \frac{1}{2s+1}\Big[Q^{(-)}_{a_1\ldots a_{2s}}\frac{1}{p_{a_{2s+1}}}$$

$$- \sum_{j=1}^{2s} Q^{(-)}_{a_1\ldots \underset{(j)}{a_{2s+1}}\ldots a_{2s}}\frac{1}{p_{a_j}}\Big] - (2s+2)\sum_{l=1}^{L_\Phi} Q^{(-)}_{a_1\ldots \underset{(j)}{a_{2s+1}} l};$$

$$P''_{a_1\ldots a_{2s+1}} = \frac{1}{2s+1}\Big[-Q^{(+)}_{a_1\ldots a_{2s}} + \sum_{j=1}^{2s} Q^{(+)}_{a_1\ldots \underset{(j)}{a_{2s+1}}\ldots a_{2s}}\Big]$$

$$- (2s+2)\sum_{l=1}^{L_\Phi} p_l Q^{(+)}_{a_1\ldots a_{2s+1} l}.$$

This completes the proof of eq. (50).

5. Conclusions

Let us recapitulate the results of the above analysis.

(a) The quantum SSG model is the only one (at least for weak coupling) in the class of two-dimensional scalar supersymmetric models with non-derivative self-interactions which is a completely integrable system (i.e., possessing an infinite number of HLQCCs).

(b) For the classical SSG model the same result is valid without any restriction on the range of the coupling constant(s).

(c) The value $\beta^2 = 4\pi$ of the quantum SSG coupling constant is a point where the theory becomes (formally) scale invariant and turns from a superrenormalizable to a non-renormalizable regime. It is not yet known to what kind of critical phenomenon (if any) this situation corresponds.

All features (a), (b), (c) have their counterparts in the usual SG model ([34], [10], [23], respectively).

The problems considered in sect. 2 were discussed for the first time in ref. [37] where the corresponding results were briefly reported.

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Note added

After submission of the present paper the author became acquainted with the preprint [38] which partially covers the material of sect. 2.

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